Published for SISSA by Springer

Received: April 8, 2014
Accepted: May 17, 2014
Published: June 17, 2014

CP violation from pure gauge in extra dimensions

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Abstract: One of the Sakharov's condition for baryogenesis is the violation of both C and CP. In the Standard Model, gauge interactions break maximally C, but CP is only broken through the Yukawa couplings in the poorly understood scalar sector. In extra-dimensional models, extra components of gauge fields behave as scalars in 4D and can acquire effective vev's through (finite) quantum effects (Hosotani mechanism). This mechanism is used to build a *toy model* with 2 extra-dimensions compactified on a flat torus T^2, where a SU(2) gauge symmetry is broken to U(1) and CP violation (in 4D) is expected. This is verified by computing a non-vanishing electric dipole moment.

Keywords: CP violation, Beyond Standard Model

ArXiv ePrint: 1403.2521

doi:10.1007/JHEP06(2014)103

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Contents

1 Introduction

In comparison with "pure gauge" theories, scalar interactions are badly understood — our ignorance being parametrized through a bunch of arbitrary (Yukawa) couplings. Moreover, while the gauge interactions are CP conserving (at least in 4D),[1] the scalars break this symmetry, but still in an arbitrary manner (through the phases of the Yukawa coefficients). The situation is well-known in the Standard Model (SM) where CP-violating freedom is only empirically constrained. It is then a sensible belief that a better insight in the scalar sector could clarify the nature of CP violation, and vice versa.

Possibly more central than CP symmetry itself is the issue of matter-antimatter asymmetry. As was pointed out by Sakharov, the emergence of a matter-antimatter asymmetry from an initially symmetrical early universe requires in 4D both C and CP violation. A more general statement would be that C and any symmetry involving C must be broken, CP being just one particular case. This is pretty much the situation we will be discussing in the present note: how C or CP invariance can be broken in theories containing only fermions and their gauge interactions. More specifically, we will discuss how C or CP conservation behave in the dimensional reduction (in the present case from 6D to 4D).

[1] We ignore mass terms which are, at least in chiral theories, a counterpart of scalar interactions.

In an attractive, though quite old idea, scalar fields are thought as spatial components of gauge fields in extra dimensions (ED) [1–3]. When extra-dimensional space is not simply connected, non trivial holonomies (or Wilson lines (WL)) can appear dynamically for non contractible cycles[2] and lead to dynamical symmetry breaking. At the level of our (3+1)-dimensional space, effective scalar fields acquire a vev, which could cause CP violation if scalar and pseudo-scalar contributions coexist. At the classical level, the WL are determined by the topology of ED and label degenerate classical vacua. The degeneracy disappears when quantum effects are taken into account, which select the physical solution. These are encoded into the effective potential for WL which depends on topology, matter content and Scherk-Schwarz (SchSch) phases (see below).

In a previous work, this idea was already used and revealed to be promising [4, 5]. One extra dimension was introduced, and the 5th components of gauge fields can yield the equivalent of pseudoscalar terms in the 4D-reduced Lagrangian, leading to a complex mass matrix and possible CP violation. Of course, this is not enough, since we can always use a chiral rotation to make them real. Therefore real masses (or in other words, half of the scalar sector) were put in by hand.[3] An appealing extension would be to add a second ED which will provide for this. This is in some way the situation we will be dealing for.

Before turning to 6D however, we should stress that this previous work viewed the Hosotani loops purely as external boundary conditions rather than dynamical variables (in the way of the Bohm-Aharonov effect). Here we will follow Hosotani's view, which sees these loops as dynamical variables, and requires the evaluation of the effective Lagrangian, beyond the tree level.

The problem proves difficult, and the present note deals with "proof of concept", namely the possibility of CP violation in 4D from pure gauge theory in 6D, but does not propose a realistic model. This is notably due to the difficulty of generating a "low mass scale", providing non-zero mass to the zero modes of the compactified theory: in the present note, we will deal either with a massless low-energy sector separated from the Kaluza-Klein scale, or accept small masses controlled by arbitrary phases in the boundary conditions.

The paper is organized as follows. In section 2 we review the notions of P, C and CP symmetries in 4D and in 6D and link them through compactification schemes. Section 3 is devoted to Hosotani mechanism which takes place when compactification implies non simply connected ED. We summarize it in the special case of the flat torus T^2 and try to include Hosotani's approach in the more modern one [8, 9]. In section 4 we use explicitly Hosotani mechanism to break CP through compactification and give simple examples in section 5. Finally, in section (6) we come back on anomaly issues which appear in chiral theories that we have neglected before. Conclusions and perspectives can be found in section 7.

[2]This can be seen (at least for abelian cases) as finite magnetic fluxes through holes in the manifold. However, these holes being outside the physical space, a flux is always ill-defined, hence the use of holonomies.

[3]We will return to this question later; in particular if complex mass terms are needed to generate CP à la Kobayashi-Maskawa, other sources of CP violation (through the Kaluza-Klein excitations for instance) remain in principle possible.

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2 P, C and CP in 4 and 6 dimensions

We use the notation γ^μ (resp. Γ^A) for 4D (resp. 6D) gamma matrices.[4] The parity transformation is given respectively by $\mathcal{P}^{-1}\psi(t,\boldsymbol{x})\mathcal{P} = \gamma^0\psi(t,-\boldsymbol{x})$ in 4D and $\mathcal{P}^{-1}\Psi(t,\boldsymbol{x})\mathcal{P} = \Gamma^0\Psi(t,-\boldsymbol{x})$ in 6D, where ψ and Ψ are 4- and 6-dimensional Dirac spinors [6]. Charge conjugation is given by $\mathcal{C}^{-1}\psi(x)\mathcal{C}^{-1} = C^{(4)}\gamma^0\psi^*(x)$ and $\mathcal{C}^{-1}\Psi(x)\mathcal{C}^{-1} = C^{(6)}\Gamma^0\psi^*(x)$ where $C^{(4)}$ (resp. $C^{(6)}$) is a matrix which satisfies ${C^{(4)}}^{-1}\gamma^\mu C^{(4)} = \pm\gamma^0\gamma^{\mu *}\gamma^0$ (resp. ${C^{(6)}}^{-1}\Gamma^A C^{(6)} = \pm\Gamma^0\Gamma^{A*}\Gamma^0$). The + sign in these relations can be used only in the absence of mass term (which is our case) and there is then an ambiguity in the definition, but we will see that this is unimportant for our purpose.

In 4 dimensions the two solutions are $C_1^{(4)} = \gamma_0\gamma_2$ and $C_2^{(4)} = \gamma_1\gamma_3$ (up to phase factors), while in 6 dimensions we find $C_1^{(6)} = \Gamma^0\Gamma^2\Gamma^4$ and $C_2^{(6)} = \Gamma^1\Gamma^3\Gamma^5$. In even dimensions, the spinors can be decomposed in two semi-spinors (or Weyl spinor) with the help of the chirality projectors[5] $P_{L/R} = \frac{1\pm\gamma^5}{2}$ (resp. $P_\pm = \frac{1\pm\Gamma^7}{2}$). Since γ^5 anticommutes with all γ's (as well as does Γ^7 with all Γ's) it is obvious that charge conjugation in 4D links ψ_L and ψ_R^* (and vice versa), while in 6D it links Ψ_+ with Ψ_+^*.[6] On the contrary, the parity connects + and − spinors in all cases (L and R in 4D). Then the CP operation which is the combination of these two connects L and L spinors in 4D, but + and − in 6D. As announced this is completely independent of the choice for $C^{(4)}$ (resp. $C^{(6)}$).

Now what does it mean? Since gauge interactions connect spinors of the same chirality, gauge symmetries give no reason to introduce both chiralities on an equal footing. Then, in all generality, P is not an automatic symmetry of gauge interactions in both 4 and 6 dimensions. However, while C symmetry is not automatic in 4D, this is always the case in 6D, and conversely for CP. For this reason we need scalar interactions in 4 dimensions to break CP (at perturbative level). In contrast if we write a theory in 6 dimensions with only (say) a + spinor then we break CP. Does it mean that the resulting effective 4D theory is not CP conserving? In other words, are the notions of CP in 4 and 6 dimensions directly related to each other? The answer is no.

To realize this we need to find a relation between 4D and 6D CP transformations. Let us focus on + spinor in 6D which is a Dirac spinor at the 4D level (with L and R components). We know that C transforms $\Psi_+(x)$ into $\Psi_+^c(x) \sim \gamma^5\gamma^2\Psi_+^*(x)$. On the other hand, + and − components being representations of the rotation group, we can use them to link $\Psi_+^c(x)$ with $\Psi_+^{CP_4}(x) \sim \gamma^0\gamma^2\Psi_+^*(t,-x_1,-x_2,-x_3,x_4',x_5')$, where (x_4',x_5') result from a rotation of (x_4,x_5). Indeed, $\Psi_+^{CP_4}(x)$ is then a CP transformation at the 4D level. One solution is to use a π-rotation in the $1-2$ and $3-5$ planes. Then $(x_4',x_5') = (x_4,-x_5)$. But any additional rotation in the $4-5$ plane leads to a valid definition.[7] Since this combination of transformations is a symmetry of the 6D theory, the 4D effective theory

[4]Our choice of representation can be found in appendix A.

[5]In 4D the + sign is identified with L and the − sign with R.

[6]This is related to the fact that in 4D (resp. 6D) ψ_L and ψ_R^* (resp. Ψ_+ and Ψ_+^*) are equivalent representations of the Lorentz group.

[7]Note that the effect of this rotation on the 4D fermion is obviously a chiral rotation.

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will be CP violating only if the compactification is incompatible with all the symmetries:

$$\begin{cases} \Psi_+ \to \Psi_+^* \\ X \equiv (x_4, x_5)^T \to \mathcal{R}\sigma^3 X = \mathcal{R}(x_4, -x_5) \equiv \hat{X} = (\hat{x}_4, \hat{x}_5). \end{cases} \tag{2.1}$$

for any rotation $\mathcal{R}$. In other words, the 4D theory will be CP violating if we fail to find a chiral rotation which reabsorbs the phases.

Let us take a simple example to illustrate this. Consider a flat torus T^2 of radii $R_4 = R_5 = R$ with the following SchSch boundary conditions (BC):[8] $\Psi(x_4 + 2\pi R, x_5) = e^{i\beta_1}\Psi(x_4, x_5)$ and $\Psi(x_4, x_5 + 2\pi R) = e^{i\beta_2}\Psi(x_4, x_5)$. Under the prescribed transformation these BC become $\Psi(\hat{x}_4 + 2\pi R\cos\theta, \hat{x}_5 + 2\pi R\sin\theta) = e^{-i\beta_1}\Psi(\hat{x}_4, \hat{x}_5)$ and $\Psi(\hat{x}_4 + 2\pi R\sin\theta, \hat{x}_5 - 2\pi R\cos\theta) = e^{-i\beta_2}\Psi(\hat{x}_4, \hat{x}_5)$. The first relation is compatible only if $\theta = \pi$ or if $\theta = 0$ and $\beta_1 \in \{0, \pi\}$, while the second one is compatible only if $\theta = 0$ or $\theta = \pi$ and $\beta_2 \in \{0, \pi\}$. Then BC break effective 4D CP symmetry as soon as β_1 and β_2 are both different from 0 and π. The result is of course independent of θ.

Note by the way that we can proceed in the same way for P and C. It is straightforward to show that P invariance requires compatibility with the transformation $X \to \mathcal{R}\sigma^3 X$, while C requires compatibility with $\Psi \to \Psi^*$ and $X \to \mathcal{R}X$. In our previous example, P is broken but not C (this leads then to CP violation).

As already mentioned in the introduction, the main point of breaking CP is to get a matter-antimatter asymmetry. Indeed even if C is broken, this is in general not enough to reach this goal. Indeed any other symmetries involving C (like CP, but CS in general) leads to matter-antimatter symmetry. In 6D the C symmetry is automatic for gauge interactions and the symmetry particle/antiparticle is respected. In 4D C is not automatic but CP leads to the same conclusion. Our idea to break this symmetry is precisely to introduce a compactification which breaks all these CS symmetries.

3 Hosotani mechanism with two ED

At the moment we work on flat space-time $\mathbb{M}^4 \times \mathbb{R}^2/G$ where the two ED are compactified by means of orbifolding[9] through one of the 17 two-dimensional space groups G [7]. These groups correspond to isometries of $\mathbb{R}^2$, which include translations, $2\pi/n$-rotations ($n = 2, 3, 4$ and 6), reflections and glide reflections.[10] These isometries must obviously be symmetries of the 6D original lagrangian. For instance, only translations and rotations can be used with a chiral lagrangian, and the possible orbifolds in this case are:[11] T^2, $T^2/\mathbb{Z}_2$, $T^2/\mathbb{Z}_3$, $T^2/\mathbb{Z}_4$ and $T^2/\mathbb{Z}_6$. We will see later that such lagrangians lead to highly non trivial issues which are due to chiral anomalies and to the interpretation of quantum corrections in ED models. Until then, we will nevertheless stick to them.

In any case, two kinds of compactification exist: the "non-magnetized" and the "magnetized" one. In the first case, a non zero field strength is unstable and the only solutions

[8] For now on Ψ means Ψ_+ unless otherwise stated.

[9] In this note, *"orbifold"* refers to any quotient spaces regardless of the existence of fixed points.

[10] Translations combined with mirror reflection.

[11] The flat torus T^2 has no fixed point and is generally not called orbifold.

are flat connections. In the second case, a non zero field strength can be stable and the solution corresponds to a physical flux orthogonal to the ED. The stability is ensured by the quantization of the flux for topological reasons [8].

Let us focus on the flat torus T^2 characterized by two radii R_4 and R_5 (we don't consider here issues of gravitational stability, and they are seen as free parameters). Because of the translation symmetry on the torus, gauge fields on this manifold must be periodic up to a gauge transformation [8, 9]:

$$A_a(y+2\pi R_i) = T_i(y)A_a(y)T_i^{-1}(y) + \frac{i}{g}T_i(y)\partial_a T_i^{-1}(y) \tag{3.1}$$

The topology of the torus requires[12] $T_4(y+2\pi R_5)T_5(y) = T_5(y+2\pi R_4)T_4(y)$.

However we must be careful, because the BC, T_i, do not fix the symmetry of the effective 4D theory. Indeed, the component of the gauge fields in the ED, playing the role of scalar fields in 4D could very well acquire a *"vev"* through quantum effects. More precisely, the ED space being multiply-connected, some non-integrable phase factors become dynamical variables which can lead to effective symmetry breaking in 4D. Indeed, it's worth stressing that neither *"vev"* nor BC are gauge invariant concepts. The true gauge invariant quantities are the so called *Wilson lines phases* defined by Hosotani as the eigenvalues of $W_{C_i}(y)T_{C_i}$, with:

$$W_{C_i}(y) = \mathcal{P}\ \exp\left(ig\int_{C_i}\mathrm{d}y'_j\langle A_j(y')\rangle\right), \tag{3.2}$$

for all the non equivalent non-contractible cycles C_i starting at y, and T_{C_i} the associated BC.

In the following we will restrict ourselves to SU(N) gauge groups for which we have an important result [8, 9]: because of the non existence of topological quantities on T^2, all stable configurations correspond to flat connexions $\langle F_{45}\rangle = 0$. In his approach [3], Hosotani takes this result as an hypothesis. Moreover, he restricts himself to homogeneous BC, i.e. $T_i(y) = T_i$. This is not mandatory, but it can help somehow to get a better insight of the physics. For this reason we first give a quick analysis of the simple case, followed by a more general, but also more technical one.

To elucidate the 4D symmetry, we are particularly interested in the zero modes (y independent) of the gauge field. Obviously these correspond to directions in the gauge group which remain unbroken after the compactification. $F_{45} = 0$ makes them satisfy $[\langle A_4\rangle, \langle A_5\rangle] = 0$. The homogeneous BC add the constraints $[\langle A_i\rangle, T_j] = 0$. In other words, $\langle A_4\rangle$ and $\langle A_5\rangle$ must be part of the Cartan subalgebra of the group. The selection of a particular solution is done at the quantum level through the so called Hosotani mechanism. Therefore, we need to compute the effective potential for A_i to find the physical symmetry. The result is of course affected by the geometry and the matter content (see section 4).

[12]This is true if we introduce fermions in a representation sensitive to the center of the group (e.g. the fundamental one). However, as long as we work with insensitive representations, the relation is valid up to an element of the center of the group [8, 9]. We neglect this at the moment.

The *"vev's"* $\langle A_4 \rangle$ and $\langle A_5 \rangle$ can be gauged away by the transformation:[13]

$$\Omega(y) = \exp\left[-ig\left(\langle A_4 \rangle y_4 + \langle A_5 \rangle y_5\right)\right], \tag{3.3}$$

and the BC matrices T_i then become[14] $T_i^{\rm sym} = \Omega(-2\pi R_i) T_i$. As previously mentioned, neither T_i nor $\langle A_i \rangle$ are physical, but only an appropriate combination. Dynamics with different T_i will give different $\langle A_i \rangle$, but the "symmetric" BC, $T_i^{\rm sym}$, obtained when the *"vev's"* are gauged away, are all equal.[15] Therefore, in all generality, we can choose $T_i = 1$ at the beginning and compute the *"vev"* $\langle A \rangle^{\rm phys}$ which contains all the physics.

Let us consider now the case where $T_i(y)$ can be y dependent. The result $\langle F_{45} \rangle = 0$ is still valid [8] and therefore the vacuum configuration for $\langle A \rangle$ must be pure gauge (this time we don't make any a priori assumption about y dependence of it):

$$\langle A_a(y) \rangle = \frac{i}{g} U(y) \partial_a U^{-1}(y),$$

where U must be compatible with the BC. If we use this expression for $\langle A_a \rangle$ into equation (3.1), it is easy to show that U must satisfy $U(y + 2\pi R_i)\partial_a U^{-1}(y + 2\pi R_i) = T_i(y) U(y) \partial_a \left(T_i(y) U(y)\right)^{-1}$ what means:

$$U(y + 2\pi R_i) = T_i(y) U(y) V_i^{-1}, \tag{3.4}$$

with V_i a constant element of the gauge group such that $[V_4, V_5] = 0$ because of the topology. For some given BC, all the classical vacua can be found by solving (3.4) for all possible V_i. Since $\langle F_{45} \rangle = 0$, we know that solutions must exist, at least for some compatible V_i. Moreover, it can be shown [8, 9] that, for SU(N) groups on T^2, solutions exist for any compatible V_i. A particular vacuum is labelled by $T_i(y)$ and $U(y)$. Now let us perform a gauge transformation U^{-1}. Then $\langle A(y) \rangle = 0$ and $T_i(y) = V_i$. Therefore, all possible classical vacua can be labelled by constant and commuting BC:

$$V_i = \exp\left(i\Theta_i\right),$$

where Θ_i are constant and commuting matrices of SU(N) algebra. Again quantum effects select the true vacuum which depends on geometry and matter content (see section 4). Let us call it $\Theta_i^{\rm phys}$. After the gauge transformation:

$$\Omega'(y) = \exp\left[-i\left(\frac{\Theta_4^{\rm phys}}{2\pi R_4} y_4 + \frac{\Theta_5^{\rm phys}}{2\pi R_5} y_5\right)\right],$$

we end up with trivial BC and a *"vev"* for the background that contains all the physics (as in the Hosotani approach):

$$\langle A_i \rangle^{\rm phys} = \frac{\Theta_i^{\rm phys}}{2\pi g R_i}.$$

[13]Since $[\langle A_4 \rangle, \langle A_5 \rangle] = 0$, Ω can be decomposed into $\Omega_4(y)\Omega_5(y) = \Omega_5(y)\Omega_4(y)$ with $\Omega_a(y) = \exp\left[-ig\langle A_a \rangle y_a\right]$.

[14]$\Omega(-2\pi R_i)$ is a shorthand notation for $\Omega(-2\pi R_4, 0)$ or $\Omega(0, -2\pi R_5)$.

[15]This is not true for all topologies. Indeed, it may be that some BC cannot be linked by any gauge transformation (3.3) for topologically satisfactory $\langle A \rangle$. It follows that we could have more than one equivalence class for BC (see for example [10]). Here, any T_i can be written as $\Omega(-2\pi R_i)$ thanks to the commutation properties and we have only one equivalence class.

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This last identification is correlated by the computation of the WL phases (3.2) in the two approaches. In the first one with trivial BC we find $W_i = \exp(i2\pi g R_i \langle A_i^{\rm phys}\rangle)$, while in the second with trivial *"vev"* we find $W_i = \exp(i\Theta_i^{\rm phys})$. Note also that it shows that the natural scale for the effective *"vev"* are the dimensions of the ED. This is expected since they are the only dimensionfull parameters.

4 CP violation induced by BC

In the last section we saw that the BC alone are not meaningful by themselves. On the other hand, we can always perform a gauge transformation that puts all the physics in the BC (in this gauge BC are identified with the WL). In all that follows we will work in this gauge. Therefore the fermionic fields[16] have the BC (to simplify notation Θ_i is identified with $\Theta_i^{\rm phys}$ defined in section 4):

$$\Psi(y + 2\pi R_i) = \exp(i\beta_i) \exp\left(i\Theta_i\right) \Psi(y),$$

with additional phases β_i allowed because fermions appear always in bilinears.[17] Note that β_i phases (or Scherk-Schwarz (SchSch) phases) are free external parameters and that we can choose them different for each fermionic field. They will enter the dynamics of fermion, possibly creating masses.

To study whether or not these BC lead to CP violation at the 4D level, we need to check their compatibility with the transformations $\Psi \to \Psi^*$ and $Y \to \hat{Y}$ (see section 2). Remember that $\hat{Y}$ can be any rotation of $(y_4, -y_5)$ (see the transformation (2.1) which makes explicit the link between C in 6D and CP in 4D) and that CP is conserved in 4D as long as we can find compatibility for one rotation. Here the gauge symmetry adds an additional freedom. Indeed, the transformations can be $\Psi \to U^*\Psi^*$, where U is any global symmetry matrix (since it keeps $\langle A\rangle = 0$).

Under the prescribed symmetries, the two BC become:[18]

$$\begin{aligned}
\Psi^{\rm CP}(y_4 + 2\pi R_4 \cos\theta, y_5 + 2\pi R_4 \sin\theta) &= \exp\left[-i\beta_4\right] \exp\left[-i\left(U\Theta_4U^{-1}\right)^*\right] \Psi^{\rm CP}(y_4, y_5) \\
\Psi^{\rm CP}(y_4 + 2\pi R_5 \sin\theta, y_5 - 2\pi R_5 \cos\theta) &= \exp\left[-i\beta_5\right] \exp\left[-i\left(U\Theta_5U^{-1}\right)^*\right] \Psi^{\rm CP}(y_4, y_5)
\end{aligned}$$

The table 1 shows the different symmetries which might be compatible with BC. The angle θ refers to the rotation $\mathcal{R}$. The columns marked β_4 and β_5 indicate a possible constraint for these phases. The next two columns show the constraints on the U matrix introduced above.[19] Note that for adjoint fermions, insensitive to the centre of the group, we have

[16] The notation refers explicitly to the fundamental representation, but it can be easily extended to the adjoint or others.

[17] We stress again that matter content plays a crucial role in the dynamics that selects the physical vacuum at quantum level. At this point, we suppose this vacuum known and encoded in the BC.

[18] It may be surprising that $\langle A\rangle$ doesn't change under the rotations. One can understand that if one remembers that the physical quantities are WL which are of course rotationally invariant.

[19] We should write $\left(U\Theta U^{-1}\right) \sim \pm\Theta^*$, but remember that $\Theta_a^\dagger = \Theta_a$, then we can use Θ_a^T instead of Θ_a^*. However, Θ_a's are diagonal (or can be diagonalized because of the topology), and therefore we can use Θ_a. Note also that these relations are not so strict. Indeed the periodicity of the exponential factor must be taken into account.

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	θ	β_4	β_5	$U\Theta_4U^{-1}$	$U\Theta_5U^{-1}$	
$R_4 \neq R_5$	0	$\{0,\pi\}$	$[0,2\pi[$	$-\Theta_4 + \frac{2\pi k}{N}T$	$\Theta_5 + \frac{2\pi k'}{N}T$	(1)
	π	$[0,2\pi[$	$\{0,\pi\}$	$\Theta_4 + \frac{2\pi k}{N}T$	$-\Theta_5 + \frac{2\pi k'}{N}T$	(2)
$R_4 = R_5$	$\pi/2$	$-\beta_5$	$-\beta_4$	$-\Theta_5 + \frac{2\pi k}{N}T$	$-\Theta_4 + \frac{2\pi k'}{N}T$	(3)
	$3\pi/2$	β_5	β_4	$\Theta_5 + \frac{2\pi k}{N}T$	$\Theta_4 + \frac{2\pi k'}{N}T$	(4)

Table 1. Hypothetical transformations that could be identified with an effective CP symmetry in 4D if compatible with boundary conditions (BC).

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a little bit more freedom. The k and k' factors take this into account for SU(N) groups ($T = \mathrm{diag}(1, \ldots, 1, 1-N)$). k and k' can take all integer values for representations which are insensitive to the centre, but must be zero in the other case.

We may expect a large variety of situations depending of the gauge group. Let us look here to some simple examples in SU(2), which we are particularly interested in (see section 5). We always have $\Theta_4 = at_3$ and $\Theta_5 = bt_3$ ($t_3 = \sigma_3/2$). Therefore, if the constraints on β's and radii are fulfilled: the transformations (1) and (2) are good candidates for CP symmetry either if a or $b = (j + k/2)\pi$, while the transformations (3) and (4) are good candidates either if $a+b$ or $a-b = (j+k/2)\pi$. k and j are integer numbers. j can always be non zero because it stands for the periodicity in the exponential factor, but k can only be non zero for representations insensitive to the centre.

There are now two main questions. (1) Which patterns can be realized (and under which conditions)? (2) At which level does CP violation manifest itself (and what could be phenomenologically promising)? As mentioned in the introduction, answering the first one is tricky because we need to compute the effective potential for WL for each group we want to study and then find the minima of this potential which depend on many parameters (SchSch phases, radii ratio, matter content). While the case of SU(2) on S^1 has been extensively studied, the behaviour for larger groups on T^2 becomes quickly hard to discuss. For the time being we focus ourselves here on simple examples. Regarding the problem of phenomenology, one of the main limitations (without any new mechanism) has been mentioned and concerns the absence of gap between light and heavy sectors. A partial answer to this issue (unfortunately quite inelegant) comes from the SchSch phases. If we choose them sufficiently small, they could account for small masses of the previously massless modes. We must however remember their influence on the dynamics of WL.

CP violation is, even in the Standard Model, a tricky issue to characterize (the Jarlskog determinants providing a partial answer). To prove that CP is violated, the safest way is to provide an "observable". Here we will deal with a single (light) fermion species and the simplest "observable" is then the electric dipole moment (EDM) of the lightest mode.[20]

[20] We study the lightest mode since we look for an understanding of CP violation at low energy. However a zero EDM for this state doesn't mean that CP is conserved (and that our previous analysis fails), as it may manifest itself at higher energy. Remember also that an EDM violates both P and CP. It is however easy to check that, with this mechanism, the 4D P symmetry is broken as soon as the CP one is.

	$\beta_5 \in [0, \pi/2]$	$\beta_5 \in [\pi/2, \pi]$	$\beta_5 \in [\pi, 3\pi/2]$	$\beta_5 \in [3\pi/2, 2\pi]$
$\beta_4 \in [0, \pi/2]$	(π, π)	$(\pi, 0)$	$(\pi, 0)$	(π, π)
$\beta_4 \in [\pi/2, \pi]$	$(0, \pi)$	$(0, 0)$	$(0, 0)$	$(0, \pi)$
$\beta_4 \in [\pi, 3\pi/2]$	$(0, \pi)$	$(0, 0)$	$(0, 0)$	$(0, \pi)$
$\beta_4 \in [3\pi/2, 2\pi]$	(π, π)	$(\pi, 0)$	$(\pi, 0)$	(π, π)

Table 2. Wilson line (WL) phases for a SU(2) theory with a 6D spinor in the fundamental representation.

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5 Examples with SU(2)

For the next examples, we will work with one of the simplest groups, i.e. SU(2). In the two first examples the matter content consists in a fermion in the fundamental (resp. the adjoint) representation. In SU(2) there are two independent dynamical variables called θ_4 and θ_5 such that $\Theta_a = \begin{pmatrix} \theta_a & 0 \\ 0 & -\theta_a \end{pmatrix}$.

The effective potential can be decomposed into [11]:

$$V_{\mathbf{eff}} = V \left(-V_{\mathbf{eff}}^{\mathrm{g+gh}} + \sum_i 2V_{\mathbf{eff}}^{\mathrm{f}_i} + \sum_i 2V_{\mathbf{eff}}^{\mathrm{ad}_i} \right),$$

where V is a positive constant, $V_{\mathbf{eff}}^{\mathrm{g+gh}}$ the contribution from gauge and ghost fields, $V_{\mathbf{eff}}^{\mathrm{f}_i}$ the contribution from fundamental fermions and $V_{\mathbf{eff}}^{\mathrm{ad}_i}$ the one from adjoint fermions. Each contribution can be written as an infinite sum over fields modes. It worth noting that this expression is only valid for Dirac spinors, and not Weyl spinors. From now on, we will use it nonetheless, and postpone the justification to the next section.

The potential must be studied numerically. The results for a theory with only fundamental fermions are simple and given in table 2.

According to [11], this result is valid for $R_4 = R_5$, but our study shows that this remains exact even for $R_4 \neq R_5$. To be more precise, the potential shape depends only on $r = R_5/R_4$. When[21] $r > 1$, the potential flattens in the y_5 direction, but the global minimum stays unchanged at least for $r \lesssim 5$. Beyond, an other local minimum becomes very close to the global one and it is hard to select the right one with numerical calculations. Nevertheless, the two candidates lead to the same phenomenological issues that we will describe here. First, it's worth noting that $\theta = 0$ and $\theta = \pi$ are particular values since then $-\theta = \theta$, $\exp[i\Theta_a] = \pm\mathbf{1}$ and the gauge symmetry remains unbroken because all SU(2) generators commute with transition functions T_i. However CP symmetry can still be broken because of the SchSch phases or $R_4 \neq R_5$ (see table 1), but another big issue is the absence of a light fermion, even with β's tuned to be small. Indeed, when β's are small, the WL are large and vice versa. More precisely one can show that the smallest "distance" between $(\beta_i + \theta_i)/2\pi$ and an integer is 0.25. Then the fermion masses

[21] The case $r < 1$ is completely symmetric.

are bounded from below (with $R_4 = R_5 = R$) $m_f > \sqrt{2}/4R \sim 0.35/R$ and there is a poor gap between the lightest mode and the KK tower.

Let us focus now on a more interesting example. Richer phenomenology can be reached if we replace the fundamental fermion by an adjoint. We will not try to give an exhaustive study of the effective potential in this case. Refs. and personal analysis show that, at least in the interesting regime $\beta_4, \beta_5 \in [0, 0.1]$ and $0.9 < r = R_5/R_4 < 1$, $(\theta_4, \theta_5) = (\pi/2, \pi/2)$. This is interesting because this time the SU(2) symmetry is spontaneously broken into U(1), and after this breaking we have a neutral fermion with mass $\sim \beta/\sqrt{2}\pi R$ which can be choose to be small. Moreover table 1 tells us that CP can be broken with non zero β's. If $r = 1$, β_4 must be different from β_5, but if $r \neq 1$ this is not even necessary. We will verify these affirmations with the EDM of our light fermion. Details about particle content and effective interactions can be found in the appendix B. The EDM is given by:[22]

$$\left|\frac{d_E R}{e^3}\right| = \left|\sum_{nm} \left\{F^+_{nm} \sin(\varphi_{3;00} - \varphi_{+;nm}) + J^+_{nm} \sin\varphi_{3;00} + K^+_{nm} \cos\varphi_{3;00}\right\}\right.$$
$$\left. + \sum_{nm} \left\{F^-_{nm} \sin(\varphi_{3;00} - \varphi_{-;-n-m}) + J^-_{nm} \sin\varphi_{3;00} + K^-_{nm} \cos\varphi_{3;00}\right\}\right|, \quad (5.1)$$

where the coefficients F, J and K and the phases $\varphi_{\pm;nm}$ and $\varphi_{3;nm}$ are functions of the θ's, the β's and r. Their explicit form can be found in the appendix C. When $r = 1$ and $\beta_4 = \beta_5$, it is easy to check that (see appendix C):

$$F^\pm_{nm} = F^\pm_{mn}; \quad J^\pm_{nm} = -K^\pm_{mn}; \quad \varphi_{\pm;nm} = \frac{\pi}{2} - \varphi_{\pm;mn}; \quad \varphi_{3;00} = \frac{\pi}{4}(3 - 2\,\mathrm{sign}(\beta)). \quad (5.2)$$

Therefore:

$$\left|\frac{d_E R}{e^3}\right| \sim (\sin\varphi_{3;00} - \cos\varphi_{3;00}) = 0.$$

This is no more true when $r \neq 1$ or $\beta_4 \neq \beta_5$. We will illustrate this with numerical evaluations. Our results can be found in table 3. We use the notation $\beta = \beta_4$, $\Delta\beta = \beta_4 - \beta_5$, $\Delta r = 1 - r$.

The behaviour of the lightest mass is easily predicted. Indeed (see appendix B) we have $m_{\text{light}} R \simeq \beta(1 + \Delta\beta/\beta + \Delta r)$, and its order of magnitude is directly related to β. On the other hand, the behaviour of the EDM is less intuitive from the analytic solutions, because of the summations and integrations in its expression. Nevertheless, we could expect a behaviour of the type:

$$\left|\frac{d_E R}{e^3}\right| \simeq C \cdot \left(\Delta r + \kappa \frac{\Delta\beta}{\beta}\right),$$

where C and κ are (almost) constant factors. Numerical evaluations show this is the case (with a pretty good accuracy) with $C \sim 10^{-2}$ and $\kappa \sim 4.5$. Obviously, the dominant CP source (Δr or $\Delta\beta/\beta$) dictates the order of magnitude for the EDM.

[22] We normalize to the scale R and the coupling constant e of the SU(2) gauge interaction.

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β	$\Delta\beta/\beta$	Δr	$m_{\text{light}}R$	$d_E R/e^3$
$[0, 10^{-1}]$	0	0	$\sqrt{2}\beta$	0
10^{-1}	0	10^{-1}	$1.35\ 10^{-1}$	$1.09\ 10^{-3}$
10^{-1}	0	10^{-2}	$1.41\ 10^{-1}$	$0.99\ 10^{-4}$
10^{-1}	0	10^{-3}	$1.41\ 10^{-1}$	$0.98\ 10^{-5}$
10^{-1}	0	10^{-4}	$1.41\ 10^{-1}$	$0.98\ 10^{-6}$
10^{-1}	10^{-1}	0	$1.35\ 10^{-1}$	$4.66\ 10^{-3}$
10^{-1}	10^{-2}	0	$1.41\ 10^{-1}$	$4.50\ 10^{-4}$
10^{-1}	10^{-3}	0	$1.41\ 10^{-1}$	$4.48\ 10^{-5}$
10^{-1}	10^{-4}	0	$1.41\ 10^{-1}$	$4.48\ 10^{-6}$
10^{-2}	10^{-1}	0	$1.35\ 10^{-2}$	$4.28\ 10^{-3}$
10^{-3}	10^{-1}	0	$1.35\ 10^{-3}$	$4.28\ 10^{-3}$
10^{-3}	10^{-1}	10^{-1}	$1.27\ 10^{-3}$	$5.71\ 10^{-3}$
10^{-3}	10^{-1}	10^{-2}	$1.33\ 10^{-3}$	$4.41\ 10^{-3}$
10^{-3}	10^{-1}	10^{-3}	$1.34\ 10^{-3}$	$4.29\ 10^{-3}$

Table 3. Numerical evaluation of the electric dipole moment (EDM) of $\psi_{3;00}$ particle (in a SU(2) theory with a 6D spinor in the adjoint representation) for different sets of parameters. The sum on (n, m) are limited to $n, m \in [-10; 10]$, and we find no big deviation from results obtained with $n, m \in [-50; 50]$. We give also its mass $m_{3;00}$ which is the lightest of the fermion spectrum. See text for more precision.

6 Chiral anomaly in 6D

Gauge theories in more than 4 space-time dimensions are not renormalizable and it could then seems dangerous to consider quantum corrections (in the effective potential for instance) in this context. However, the WL-dependent part of V_{eff} turns out finite, at least at the one loop level, and can then be evaluated unambiguously [12]. For chiral theory, however, things become worse, because of the presence of anomalies. In 6D, they come from the square diagram[23] (equivalent to triangle diagram in 4D) which certainly plays a role in the effective potential.

Concretely, anomalies originate from UV divergences, but they are finite and calculable IR effects (even in more than 4D) which then do not depend on the UV completion of the theory [13, 14]. For non renormalizable theories, which are only valid under a certain energy scale, anomalies can cancel among themselves (like in 4D), but they can also cancel with effects originating from an unknown UV sector. It is not our point to discuss these issues here, and we will avoid them with the introduction of both 6D chiralities (+ and −) in the same representation. However, we will use the BC to differentiate the masses of the

[23] In non abelian theories, there exists other pathological diagrams, but they can be related to this one through gauge invariance.

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light excitations of these fields. As announced, this will lead to small modifications in the effective potential, but our previous results will remain intact.

The only modification of $V_{\mathbf{eff}}$ appears in the fermion contributions. We must do the replacements:

$$\begin{aligned} 2V_{\mathbf{eff}}^{\mathrm{f}_i}(\beta_4, \beta_5) &\longrightarrow V_{\mathbf{eff}}^{\mathrm{f}_i}(\beta_{4+}, \beta_{5+}) + V_{\mathbf{eff}}^{\mathrm{f}_i}(\beta_{4-}, \beta_{5-}) \\ 2V_{\mathbf{eff}}^{\mathrm{ad}_i}(\beta_4, \beta_5) &\longrightarrow V_{\mathbf{eff}}^{\mathrm{ad}_i}(\beta_{4+}, \beta_{5+}) + V_{\mathbf{eff}}^{\mathrm{f}_i}(\beta_{4-}, \beta_{5-}). \end{aligned}$$

We have checked numerically that, in the range of SchSch phases we work with, this keeps the minimum of the total effective potential at $(\theta_4, \theta_5) = (\pi/2, \pi/2)$. This is why we used it in section 5, even if at that time, we had only introduced one chirality.[24]

It is easy to verify that in the case of degenerate SchSch phases for $+$ and $-$ chiralities, the EDMs are exactly opposite for the two sectors, thus restoring CP ("CP doubling"). To be more precise, the lightest modes could still be distinguished through their different couplings, but we prefer to provide a case where CP violation is explicit in terms of low energy parametrization. This is easily obtained if we choose different SchSch phases for $+$ and $-$ chiralities. As a simple class of examples, let us take $\Delta r = 0$, $\Delta\beta_- = 0$ and $\Delta\beta_+ \neq 0$. In this way, in the "$-$" sector $d_E = 0$ while in the "$+$" sector $d_E \neq 0$.

7 Conclusion and perspectives

We made use of the Hosotani mechanism to generate both gauge and CP symmetry breaking through compactification from a 6-dimensional model. Though we found examples where it works, our solutions is far from being realistic, and they must be seen more as *"proof of concept"*. One of the major difficulty of the work is the high level of entanglement in the approach. Indeed, the final result depends both on matter content (representations), BC (SchSch phases) and WL phases, while the latter depend in turn on the formers and are dynamically determined through a potential which must be numerically evaluated.

The next steps in this program should be the resolution of the two main drawbacks of the present solutions. First new compactification mechanism (like orbifold or flux compactifications[25]) might be employed to reach a chiral theory in 4D (at this point the only difference between left and right couplings in the gauge sector comes through a phase). Moreover, we'd like to avoid the presence of two (nearly) identical fermionic sectors without introducing anomalies in the theory. Secondly (but this maybe even more ambitious), a mechanism which produces a low energy sector naturally separated from the Kaluza-Klein scale would be very welcome. For instance, in more complex situations, one can hope for an effective low energy potential between the remaining scalars, what would provide the lower mass scale, but this goes beyond this "proof of concept" paper.

[24] Note by the way that the same conclusion holds for fundamental representation and our previous results stay qualitatively identical. There is no way to force θ values to be different from 0 or π which lead to unbroken gauge symmetry. Moreover, the minimum always arranges to prevent small masses in the spectrum.

[25] See for instance interesting application for gauge symmetry breaking in [15].

Acknowledgments

We are indebted to E. Nugaev and S. Troitsky for helpful discussions. M.L. thanks the Service de Physique Théorique at Université Libre de Bruxelles for kind hospitality. This work is funded in part by IISN and by Belgian Science Policy (IAP VII/37). This work has been supported in part by the grant of the President of the Russian Federation NS-2835.2014.2 (M.L.).

A Dirac matrices in 6D

We use the following representation for the Dirac matrices in 6D:

$$\Gamma^A = \begin{pmatrix} 0 & \Sigma^A \\ \bar{\Sigma}^A & 0 \end{pmatrix} \quad \text{and} \quad \Gamma^7 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $\Sigma^\mu = \gamma^0\gamma^\mu$, $\Sigma^4 = i\gamma^0\gamma^5$, $\Sigma^5 = \gamma^0$ and $\bar{\Sigma}^0 = \Sigma^0$, $\bar{\Sigma}^{A\neq 0} = -\Sigma^{A\neq 0}$. The γ's are 4D Dirac matrices:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix} \quad \text{and} \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $\sigma_\mu = (1, \sigma_i)$ and $\bar{\sigma}_\mu = (1, -\sigma_i)$.

B Effective 4D theory for an SU(2) adjoint fermion

Not considering here the anomalies, we work only with a 6D Weyl fermion Ψ in the adjoint representation of SU(2) and a gauge field A_A. These fields can be decomposed in the Cartan basis $\{T_+, T_-, T_3\}$ which satisfies $[T_+, T_-] = T_3$ and $[T_2, T_\pm] = \pm T_\pm$. The 6D lagrangian can be written

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}[F_{AB}F^{AB}] + 2\ \mathrm{Tr}[i\Psi^\dagger\bar{\Sigma}^A D_A\Psi],$$

with $F_{AB} = \partial_A A_B - \partial_B A_A - ie[A_A, A_B]$, $D_A = \partial_A - ie[A_A, \bullet]$ the covariant derivative and $\bar{\Sigma}^A = \gamma^0 \cdot \{\gamma^0, -\gamma^i, -i\gamma^5, -1\}$. If we define $\psi = \gamma^0\Psi$ we can write the fermionic part of the lagrangian in the following form:

$$\begin{aligned}
\mathcal{L} \supset\ & i\bar{\psi}_+\gamma^\mu\partial_\mu\psi_+ + i\bar{\psi}_-\gamma^\mu\partial_\mu\psi_- + i\bar{\psi}_3\gamma^\mu\partial_\mu\psi_3 \\
& - i\bar{\psi}_+(\partial_5 - i\gamma^5\partial_4)\psi_+ - i\bar{\psi}_-(\partial_5 - i\gamma^5\partial_4)\psi_- - i\bar{\psi}_3(\partial_5 - i\gamma^5\partial_4)\psi_3 \\
& + e\bar{\psi}_+\gamma^\mu(A_{3;\mu}\psi_+ - A_{+;\mu}\psi_3) + e\bar{\psi}_-\gamma^\mu(A_{-;\mu}\psi_3 - A_{3;\mu}\psi_-) + e\bar{\psi}_3\gamma^\mu(A_{+;\mu}\psi_- - A_{-;\mu}\psi_+) \\
& - e\bar{\psi}_+(A_{3;5}\psi_+ - A_{+;5}\psi_3) - e\bar{\psi}_-(A_{-;5}\psi_3 - A_{3;5}\psi_-) - e\bar{\psi}_3(A_{+;5}\psi_- - A_{-;5}\psi_+) \\
& + e\bar{\psi}_+ i\gamma^5(A_{3;4}\psi_+ - A_{+;4}\psi_3) + e\bar{\psi}_- i\gamma^5(A_{-;4}\psi_3 - A_{3;4}\psi_-) + e\bar{\psi}_3 i\gamma^5(A_{+;4}\psi_- - A_{-;4}\psi_+)
\end{aligned} \tag{B.1}$$

To find the 4D effective lagrangian we need to decompose ψ and A_A into fundamental modes which satisfy BC. For an adjoint fermion these are given by:

$$\psi(y + 2\pi R_i) = e^{i\beta_i}e^{i\theta_i T_3}\psi(y)e^{-i\theta_i T_3},$$

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or, in the Cartan basis:

$$\begin{cases} \psi_3(y+2\pi R_i) = e^{i\beta_i}\psi_3(y) \\ \psi_\pm(y+2\pi R_i) = e^{i(\beta_i\pm\theta_i)}\psi_\pm(y). \end{cases}$$

Therefore the (normalized) mode decompositions are:

$$\begin{cases} \psi_3(y) = \dfrac{1}{2\pi\sqrt{R_4R_5}}\displaystyle\sum_{nm} e^{i\left(n+\frac{\beta_4}{2\pi}\right)\frac{y_4}{R_4}} e^{i\left(m+\frac{\beta_5}{2\pi}\right)\frac{y_5}{R_5}}\psi_{3;nm} \\ \psi_\pm(y) = \dfrac{1}{2\pi\sqrt{R_4R_5}}\displaystyle\sum_{nm} e^{i\left(n+\frac{\beta_4\pm\theta_4}{2\pi}\right)\frac{y_4}{R_4}} e^{i\left(m+\frac{\beta_5\pm\theta_5}{2\pi}\right)\frac{y_5}{R_5}}\psi_{\pm;nm}. \end{cases}$$

The decompositions for A_A are obtained with $\beta_4 = \beta_5 = 0$.

Let us introduce these decompositions in (B.1). The first line gives the kinetic energy for each mode. The second line gives the effective 4D masses:

$$\begin{cases} m_{3;nm} = -\dfrac{1}{R}\left[\left(m+\dfrac{\beta_5}{2\pi}\right) - i\gamma^5 r\left(n+\dfrac{\beta_4}{2\pi}\right)\right] \\ m_{\pm;nm} = -\dfrac{1}{R}\left[\left(m+\dfrac{\beta_5\pm\theta_5}{2\pi}\right) - i\gamma^5 r\left(n+\dfrac{\beta_4\pm\theta_4}{2\pi}\right)\right]. \end{cases}$$

To get real (and positive) masses, we perform a chiral rotation $\psi \to e^{i\frac{\varphi}{2}\gamma^5}\psi$, where the phases are given by:

$$\begin{cases} \exp[i\varphi_{3;nm}] = -\dfrac{\left(m+\frac{\beta_5}{2\pi}\right)+ir\left(n+\frac{\beta_4}{2\pi}\right)}{\sqrt{\left(m+\frac{\beta_5}{2\pi}\right)^2+r^2\left(n+\frac{\beta_4}{2\pi}\right)^2}} \\ \exp[i\varphi_{\pm;nm}] = -\dfrac{\left(m+\frac{\beta_5\pm\theta_5}{2\pi}\right)+ir\left(n+\frac{\beta_4\pm\theta_4}{2\pi}\right)}{\sqrt{\left(m+\frac{\beta_5\pm\theta_5}{2\pi}\right)^2+r^2\left(n+\frac{\beta_4\pm\theta_4}{2\pi}\right)^2}}. \end{cases} \tag{B.2}$$

The real masses are then:

$$\begin{cases} m_{3;nm} = \dfrac{1}{R}\sqrt{\left(m+\dfrac{\beta_5}{2\pi}\right)^2+r^2\left(n+\dfrac{\beta_4}{2\pi}\right)^2} \\ m_{\pm;nm} = \dfrac{1}{R}\sqrt{\left(m+\dfrac{\beta_5\pm\theta_5}{2\pi}\right)^2+r^2\left(n+\dfrac{\beta_4\pm\theta_4}{2\pi}\right)^2}. \end{cases}$$

Note that the two last relations (5.2), valid for $\beta_4 = \beta_5$ and $r = 1$ can be easily proven here with the definitions (B.2). If we remind that the effective potential imposes $\theta_4 = \theta_5$, we see that the exchange of n and m in these relations is equivalent to the exchange of real and imaginary part of the phases, what means $\varphi_{\pm;nm} = \pi/2 - \varphi_{\pm;mn}$. Finally $\varphi_{3;00} = \pm\pi/2$, since in this case it has its real and imaginary parts equal. The sign is determined by the sign of β, and the solution can be written synthetically as $\varphi_{3;00} = \frac{\pi}{4}(3 - 2\mathrm{sign}(\beta))$.

The third line in (B.1) gives the effective interactions with 4D vector bosons, while the fourth and fifth ones give the interactions with 4D scalars bosons. To get an interesting

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form we need to perform the chiral rotation, but also to go in the mass eigenbasis for the bosons. To study this let us have a look to the quadratic part of the gauge lagrangian:

$$\begin{aligned}\mathcal{L} \supset & -\frac{1}{4}\left(\partial_\mu A_{3;\nu} - \partial_\nu A_{3;\mu}\right)^2 + \frac{1}{2}(\partial_\mu A_{3;4})^2 + \frac{1}{2}(\partial_\mu A_{3;5})^2 + \frac{1}{2}(\partial_4 A_{3;\mu})^2 + \frac{1}{2}(\partial_5 A_{3;\mu})^2 \\ & -\frac{1}{2}(\partial_4 A_{3;5})^2 - \frac{1}{2}(\partial_5 A_{3;4})^2 + (\partial_4 A_{3;5})(\partial_5 A_{3;4}) - (\partial_4 A_{3;\mu})(\partial^\mu A_{3;4}) - (\partial_5 A_{3;\mu})(\partial^\mu A_{3;5}) \\ & -\frac{1}{2}\left|\partial_\mu A_{+;\nu} - \partial_\nu A_{+;\mu}\right|^2 + |\partial_\mu A_{+;4}|^2 + |\partial_\mu A_{+;5}|^2 + |\partial_4 A_{+;\mu}|^2 + |\partial_5 A_{+;\mu}|^2 \\ & -|\partial_4 A_{+;5}|^2 - |\partial_5 A_{+;4}|^2 + \left[(\partial_4 A^*_{+;5})(\partial_5 A_{+;4}) - (\partial_4 A^*_{+;\mu})(\partial^\mu A_{+;4}) - (\partial_5 A^*_{+;\mu})(\partial^\mu A_{+;5}) + h.c.\right]\end{aligned} \tag{B.3}$$

From BC we can convert ∂_4, ∂_5 into mass matrices for the vector and scalar bosons. The vector bosons $A_{3;\mu,nm}$ (resp. $A_{+;\mu,nm}$) have masses $M_{3;nm}$ (resp. $M_{+;nm}$) given by:

$$\begin{cases} M_{3;nm} = \dfrac{1}{R}\sqrt{m^2 + r^2 n^2} \\ M_{+;nm} = \dfrac{1}{R}\sqrt{\left(m + \dfrac{\theta_5}{2\pi}\right)^2 + r^2\left(n + \dfrac{\theta_4}{2\pi}\right)^2}. \end{cases}$$

One of the bosons in the spectrum ($A_{3;\mu,00}$) remains massless as expected by the symmetry breaking pattern. On the other hand $A_{+;\mu,00}$ acquires a mass through the Hosotani mechanism. It is worth noting that, except for $A_{3;\mu,00}$, all the 4D vector bosons can be expressed in terms of complex fields.

In the scalar sector, there is a mixing between A_4 and A_5. The mass matrices are given by:

$$\begin{cases} \begin{pmatrix} A^*_{3;4,nm} & A^*_{3;5,nm}\end{pmatrix} \left[\dfrac{1}{R^2}\begin{pmatrix} m^2 & -rnm \\ -rnm & r^2n^2 \end{pmatrix}\right] \begin{pmatrix} A_{3;4,nm} \\ A_{3;5,nm}\end{pmatrix} \\ \begin{pmatrix} A^*_{+;4,nm} & A^*_{+;5,nm}\end{pmatrix} \left[\dfrac{1}{R^2}\begin{pmatrix} \left(m+\frac{\theta_5}{2\pi}\right)^2 & -r\left(n+\frac{\theta_4}{2\pi}\right)\left(m+\frac{\theta_5}{2\pi}\right) \\ -r\left(n+\frac{\theta_4}{2\pi}\right)\left(m+\frac{\theta_5}{2\pi}\right) & r^2\left(n+\frac{\theta_4}{2\pi}\right)^2 \end{pmatrix}\right] \begin{pmatrix} A_{+;4,nm} \\ A_{+;5,nm}\end{pmatrix} \end{cases}$$

The mass eigenstates $g_{3;nm}$ and $g_{+;nm}$ are massless, while $h_{3;nm}$ and $h_{+;nm}$ have masses $M_{3;nm}$ and $M_{+;nm}$. They are given by:

$$\begin{cases} g_{3;nm} = \dfrac{mA_{3;5,nm} + rnA_{3;4,nm}}{\sqrt{m^2 + r^2n^2}} \\ h_{3;nm} = \dfrac{mA_{3;4,nm} - rnA_{3;5,nm}}{\sqrt{m^2 + r^2n^2}} \\ g_{+;nm} = \dfrac{\left(m+\frac{\theta_5}{2\pi}\right)A_{+;5,nm} + r\left(n+\frac{\theta_4}{2\pi}\right)A_{+;4,nm}}{\sqrt{\left(m+\frac{\theta_5}{2\pi}\right)^2 + r^2\left(n+\frac{\theta_4}{2\pi}\right)^2}} \\ h_{+;nm} = \dfrac{\left(m+\frac{\theta_5}{2\pi}\right)A_{+;4,nm} - r\left(n+\frac{\theta_4}{2\pi}\right)A_{+;5,nm}}{\sqrt{\left(m+\frac{\theta_5}{2\pi}\right)^2 + r^2\left(n+\frac{\theta_4}{2\pi}\right)^2}}. \end{cases}$$

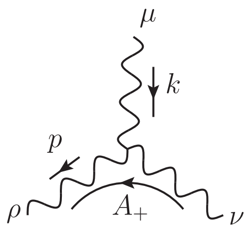

Figure 1. $e(p_\nu g_{\mu\rho} + p_\rho g_{\mu\nu} - 2p_\mu g_{\nu\rho} + k_\nu g_{\mu\rho} + k_\mu g_{\nu\rho} - 2k_\rho g_{\mu\nu})$.

If we perform a rotation toward the mass eigenbasis in (B.3), we find that g scalar bosons play the role of goldstone bosons. They are eaten by the vector bosons which acquire masses. The only physical goldstone boson is $g_{3;00}$. Actually, $h_{3;00}$ is massless too and the effective theory contains two massless scalar degrees of freedom, what could be a drawback.

We can now find all the interaction terms in the right basis. In addition to the fermion-fermion-vector and fermion-fermion-scalar interactions, we have still a bunch of vector-scalar interactions implying 3 or 4 particles. We will not write all of them but focus ourselves on the one participating in the one loop diagrams for the EDM. These are the 3 particles interactions with at least one $A_{3;\mu,00}$ boson (the external "photon"). They come from the following part of the 6D lagrangian:

$$\begin{aligned}\mathcal{L} \supset\ & ie\left[A_+^\nu A_+^{*\mu}\partial_\nu A_{3;\mu} - A_{+;4}A_+^{*\mu}\partial_4 A_{3;\mu} - A_{+;5}A_+^{*\mu}\partial_5 A_{3;\mu}\right] \\ & + ie\left[\left(\partial_\nu A_{+;\mu} - \partial_\mu A_{+;\nu}\right)A_+^{*\nu}A_3^\mu - \left(\partial_4 A_{+;\mu} - \partial_\mu A_{+;4}\right)A^*_{+;4}A_3^\mu - \left(\partial_5 A_{+;\mu} - \partial_\mu A_{+;5}\right)A^*_{+;5}A_3^\mu\right] + h.c.\end{aligned}$$

Let us now introduce the mode decompositions (with only the mode (00) for $A_{3;\mu}$) to yield:

$$\begin{aligned}\mathcal{L} \supset\ & ie\ A^{*\mu}_{+;nm}A^\nu_{+;nm}\partial_\nu A_{3;\mu,00} + ie\left(\partial_\nu A_{+;\mu,nm} - \partial_\mu A_{+;\nu,nm}\right)A^{*\nu}_{+;nm}A^\mu_{3;00} \\ & + eM_{+;nm}\ g^*_{+;nm}A_{+;\mu,nm}A^\mu_{3,00} + ie\ g^*_{+;nm}\partial_\mu g_{+;nm}A^\mu_{3;00} + ie\ h^*_{+;nm}\partial_\mu h_{+;nm}A^\mu_{3;00} + h.c.\end{aligned}$$

We give all the corresponding diagrams below (figures 1 to 7). The additional diagrams are for the fermion-fermion-vector or fermion-fermion-scalar interactions implying at least one $A_{3;\mu,00}$ boson or a $\psi_{3;00}$ fermion. We use simplified notations: $\psi_3 = \psi_{3;00}$, $A_+ = A_{+;\mu,nm}$, $\psi_\pm = \psi_{\pm;\pm n\pm m}$, $g_+ = g_{+;nm}$, $h_+ = h_{+;nm}$, $M_+ = M_{+;nm}$, $\varphi = \varphi_{3;00}$, $\varphi_\pm = \varphi_{\pm;\pm n\pm m}$ and $\varphi^0_\pm = \varphi_{\pm;\pm n\pm m}(\beta_4 = \beta_5 = 0)$. Finally, all the wiggled lines without label are $A_{3;\mu;00}$ "photons". Note that charge conservation combined with momentum conservation in ED imposes $A_{+;nm}$ interacts with $\psi_{3;00}$ and either $\psi_{+;nm}$ or $\psi_{-;-n-m}$.

C Electric dipole moment of $\psi_{3;00}$

Six kinds of diagrams are involved in the one loop evaluation of the EDM for $\psi_{3;00}$. We show them for a $\psi_{+;nm}$ in the loop in figure 8. For the $\psi_{-;-n-m}$, the fields A_+, g_+ and h_+ must be replaced by complex conjugate fields (or the arrows reversed). At the end we must sum up the + and − contributions and sum over all nm modes.

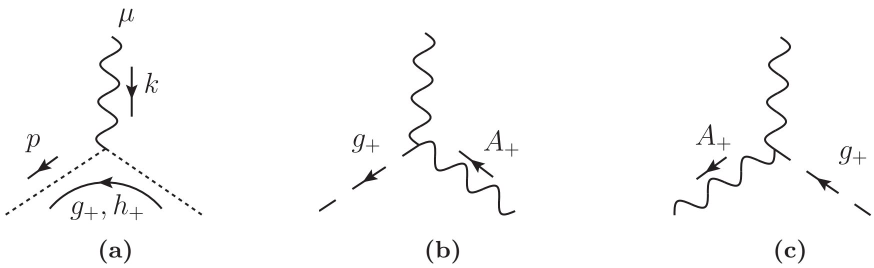

Figure 2. (a) $= e(2p_\mu - k_\mu)$; (b) and (c) $= eM_+$.

Figure 3. (a) $= -e \exp\left[-i(\varphi - \varphi_+)\gamma^5/2\right] \gamma^\mu$; (b) $= -e \exp\left[i(\varphi - \varphi_+)\gamma^5/2\right] \gamma^\mu$.

Figure 4. (a) $= e \exp\left[-i(\varphi - \varphi_-)\gamma^5/2\right] \gamma^\mu$; (b) $= e \exp\left[i(\varphi - \varphi_-)\gamma^5/2\right] \gamma^\mu$.

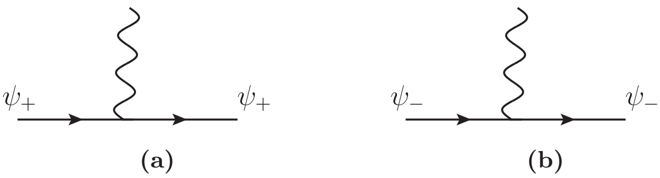

Figure 5. (a) $= e\gamma^\mu$; (b) $= -e\gamma^\mu$.

The contributions to F_{nm} come from diagrams 8a and 8c. The diagrams 8b and 8d give no contributions. Finally the contributions to J_{nm} and K_{nm} come from diagrams 8e and 8f.

$$
\begin{aligned}
F^{\pm}_{nm} &= \pm \frac{\hat{m}_{\pm;\pm n \pm m}}{(4\pi)^2} \left[4 \int_0^1 \mathrm{d}x \, \frac{x(1-x)}{\Delta_{\pm;nm}(x)} + 3 \int_0^1 \mathrm{d}x \, \frac{(1-x)^2}{\tilde{\Delta}_{\pm;nm}(x)} \right] \\
J^{\pm}_{nm} &= \pm \frac{R \cos \varphi^0_{\pm}}{(4\pi)^2} \int_0^1 \mathrm{d}x \, \frac{(1-x)}{\tilde{\Delta}_{\pm;nm}(x)} \\
K^{\pm}_{nm} &= \mp \frac{R \sin \varphi^0_{\pm}}{(4\pi)^2} \int_0^1 \mathrm{d}x \, \frac{(1-x)}{\tilde{\Delta}_{\pm;nm}(x)},
\end{aligned}
$$

where the functions $\Delta_{\pm;nm}(x)$ and $\tilde{\Delta}_{\pm;nm}(x)$ are polynomials given by:

$$
\Delta_{\pm;nm}(x) = \hat{m}^2_{3;00} \, x^2 + (\hat{M}^2_{+;nm} - \hat{m}^2_{\pm;\pm n \pm m} - \hat{m}^2_{3;00}) \, x + \hat{m}^2_{\pm;\pm n \pm m}
$$

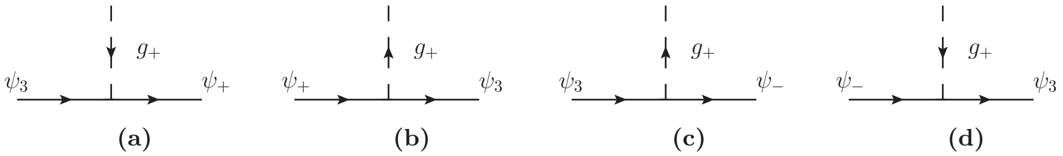

Figure 6. (a) and (b) $= -e \exp\left[i\left(\frac{\varphi+\varphi_+-2\varphi_+^0}{2}\right)\gamma^5\right]$; (c) and (d) $= -e \exp\left[i\left(\frac{\varphi+\varphi_--2\varphi_-^0}{2}\right)\gamma^5\right]$.

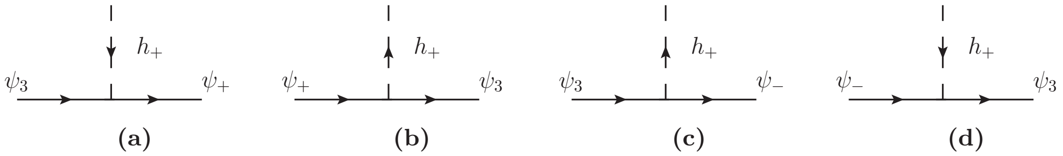

Figure 7. (a) and (b) $= e \exp\left[i\left(\frac{\varphi+\varphi_+-2\varphi_+^0+\pi}{2}\right)\gamma^5\right]$; (c) and (d) $= e \exp\left[i\left(\frac{\varphi+\varphi_--2\varphi_-^0+\pi}{2}\right)\gamma^5\right]$.

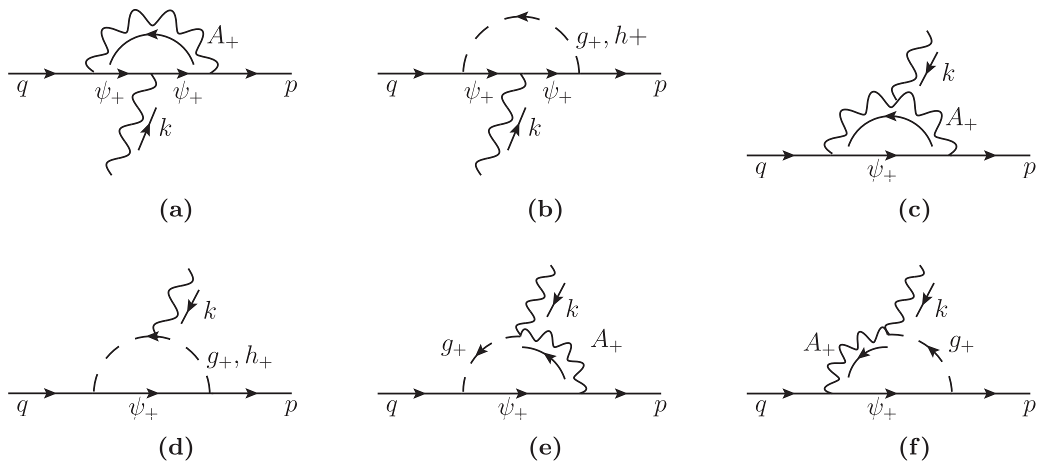

Figure 8. One-loop contributions to EDM for $\psi_{3;00}$. Contributions with $\psi_{-;-n-m}$ in the loop must be included as well.

$$\tilde{\Delta}_{\pm;nm}(x) = \hat{m}_{3;00}^2\ x^2 - (\hat{M}_{+;nm}^2 - \hat{m}_{\pm;\pm n\pm m}^2 + \hat{m}_{3;00}^2)\ x + \hat{M}_{+;nm}^2,$$

and all the “hat masses” ($\hat{m}$, ...) are the dimensionless masses Rm (the R factor being factorized in the expression of d_E).

It is now easy to check the two first relations (5.2). To compute $F_{mn}^{\pm}$, we must exchange n and m in all the masses $m_{\pm;\pm n\pm m}$ and $M_{+;nm}$. Since $\theta_4 = \theta_5$ (imposed by the effective potential), and $\beta_4 = \beta_5$ (imposed by hand as an hypothesis), these stay unchanged and F as well. In the same way $J_{nm}^{\pm}$ and $K_{nm}^{\pm}$ do not change through $\tilde{\Delta}_{\pm;nm}$ but only through the phase $\varphi_{\pm}^0$. The expressions (B.2) with $\theta_4 = \theta_5$ ($\beta_4 = \beta_5 = 0$ by definition) show that exchanging n and m is equivalent to exchanging real and imaginary part, or $\cos\varphi_{\pm}^0$ and $\sin\varphi_{\pm}^0$. We then conclude easily that $J_{mn}^{\pm} = -K_{nm}^{\pm}$.

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D + and − chiralities sector

The only difference between + and − chirality lagrangians is the matrices used to form a 6D vectors that can be "contracted" with covariant derivative. For + these are $\bar{\Sigma}^A$ matrices defined in appendix B, and for − these are Σ^A matrices, defined as $\Sigma^0 = \bar{\Sigma}^0$ and $\Sigma^{A\neq 0} = -\bar{\Sigma}^{A\neq 0}$. At the 4D level, an other difference appears because $\Psi_+ \sim \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$, while $\Psi_- \sim \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$. To form the usual Dirac kinetic terms, we have to rewrite the lagrangians in terms of the matrices $\bar{\Sigma}^A\gamma^0 = (\gamma^\mu, i\gamma^5, -1)$ and $\gamma^0\Sigma^A = (\gamma^\mu, i\gamma^5, 1)$. Thus, the only remaining difference, is a sign in the fifth component of the covariant derivative D_5. This doesn't change the mass spectrum, but only the chiral phases and the interactions with A_5 bosons (see appendix B for more details). This is equivalent to change sign of m, β_5 and θ_5 in (B.2), sign of $\varphi^0_\pm$ and e in diagrams of figure 6 (interaction with g_+ bosons) and sign of $\varphi^0_\pm$ in diagrams of figure 7 (interaction with h_+ boson). Now we see that $F^\pm_{nm}$ and $J^\pm_{nm}$ don't change, while $K^\pm_{nm}$ changes sign. But we must not forget that $J^\pm_{nm}$ and $K^\pm_{nm}$ come from interaction with one g_+ boson (see appendix C), then they undergo an additional change of sign. We have then the following transformations:

$$\begin{aligned}
F^\pm_{nm}\sin(\varphi_{3;00} - \varphi_{\pm;\pm n \pm m}) &\longrightarrow F^\pm_{nm}\left(-\sin(\varphi_{3;00} - \varphi_{\pm;\pm n \pm m})\right) \\
J^\pm_{nm}\sin\varphi_{3;00} &\longrightarrow \left(-J^\pm_{nm}\right)\sin\varphi_{3;00} \\
K^\pm_{nm}\cos\varphi_{3;00} &\longrightarrow K^\pm_{nm}\left(-\cos\varphi_{3;00}\right),
\end{aligned}$$

which lead to the conclusion that d_E changes sign (see (5.1)).

Therefore, if we choose the same SchSch phases for + and − chiralities, we end up with two fermionic sectors (let us call them P- and M-sectors, which interact only through gauge and scalar interactions) with exactly the same mass spectra, and with an equal and opposite EDM for the two lightest modes.

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